# Slow Fibonacci Walks

Sam Spiro, UC San Diego.

Joint with Fan Chung and Ron Graham.

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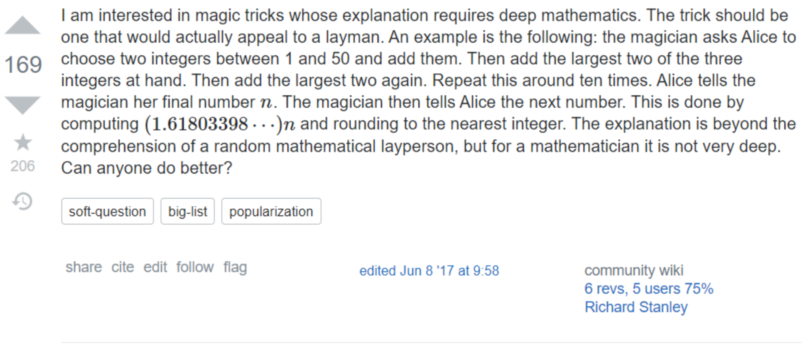

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Given positive integers $a_1, a_2$, we define the $(a_1, a_2)$-Fibonacci walk to be the sequence $w_k = w_k(a_1, a_2)$ satisfying

$$w_1 = a_1, \; w_2 = a_2, \; w_{k+2} = w_{k+1} + w_k.$$

For example, if $w_k = w_k(10, 2)$, this gives the sequence

$$10, \; 2, \; 12, \; 14, \; 26, \; 40, \; 66 \ldots$$

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For example, if $w_k = w_k(10, 2)$, this gives the sequence

$$10,\ 2,\ 12,\ 14,\ 26,\ 40,\ 66\ldots$$

We say that $w_k$ is an $n$-Fibonacci walk if $w_s = n$ for some $s$. For example, the above $w_k$ is a 40-Fibonacci walk.

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For example, the following are all 40-Fibonacci walks.

$$1024,\ \underline{40},\ 1064\ldots$$
$$8,\ 8,\ 16,\ 24,\ \underline{40},\ 64\ldots$$
$$5,\ 5,\ 10,\ 15,\ 25,\ \underline{40},\ 65\ldots$$
$$10,\ 2,\ 12,\ 14,\ 26,\ \underline{40},\ 66\ldots$$

However, the first two can't be slow (since the next two achieve 40 with $s = 6$), and one can verify that $w_k(5, 5)$ and $w_k(10, 2)$ are (the unique) 40-slow Fibonacci walks.

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**Lemma**

*$s(n) = 2$ iff $n = 1$, in which case $(x, 1)$ is a 1-good pair for all $x$.*

$w_3(x, y) = x + y \geq 2$, $w_3(1, n-1) = n$. □

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**Lemma**

*Assume $s(n) = s > 2$. If $(b, a)$ is $n$-good, then $a \leq b$.*

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$$4, \; 10, \; 14, \cdots$$

we should start

$$6, \; 4, \; 10, \; 14, \cdots$$

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$w_k(b, a) = af_{k-1} + bf_{k-2}$, *with* $f_k$ *the Fibonacci numbers.*

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*Let $s = s(n) > 2$. If $(b, a)$ is n-good, then $(b', a')$ is n-good iff $a' = a + kf_{s-2} \geq 1$ and $b' = b - kf_{s-1} \geq 1$ for some $k$.*

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For example, if we know $s(40) = 6$ and $(10,2)$ is 40-good, then so is $(10 - kf_5, 2 + kf_4) = (10 - 5k, 2 + 3k)$, which only makes sense if $k = 0, 1$.

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**Proof.**

By the above lemma, every $n$-good pair is a solution to the diophantine equation $n = w_s(b',a') = a'f_{s-1} + b'f_{s-2}$, and the result follows since $\gcd(f_{s-1}, f_{s-2}) = 1$ for $s > 2$. □

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**Theorem (Englund, Bicknell-Johnson (1997); Jones, Kiss (1998); Chung, Graham, S. (2019))**

*For* $n > 1$ *with* $s = s(n)$*, there exist unique integers* $a = a(n),\ b = b(n)$ *such that* $n = af_{s-1} + bf_{s-2}$ *and* $1 \leq a \leq b \leq f_{s-1}$*. In this case* $(b, a)$ *is* $n$*-good.*

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Because $w_k(x, y) = yf_{k-1} + xf_{k-2}$, by definition of $s$ there exist $a', b'$ such that $n = w_s(b', a') = a' f_{s-1} + b' f_{s-2}$.

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**Theorem (Chung, Graham, S. (2019))**

*Given $n$, let $c, p$ be such that $n = \frac{1}{\sqrt{5}} c\phi^p$ with $\frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}}\phi$. Then*

$$T(n) = \begin{cases} \frac{1}{2\sqrt{5}\phi^4 c} + O(n^{-1/2}) & p \equiv 1 \mod 2, \\ \frac{\sqrt{5}}{2}c + \frac{1+\phi^{-5}}{2\sqrt{5}c} - 1 + O(n^{-1/2}) & p \equiv 0 \mod 2,\ c \leq \frac{1+\phi^{-3}}{\sqrt{5}}, \\ 1 - \frac{\sqrt{5}}{2}\phi^{-1}c - \frac{1+\phi^{-2}}{2\sqrt{5}c} + O(n^{-1/2}) & p \equiv 0 \mod 2,\ c \geq \frac{1+\phi^{-3}}{\sqrt{5}}. \end{cases}$$

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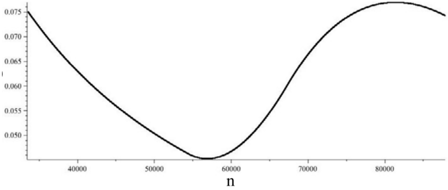
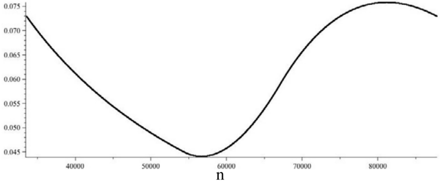



(a) Data plot of $T(n)$.




(b) Theory plot of $T(n)$.

Note that this value oscillates.

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## Corollary

*For any $n$ with $s(n) = s > 2$, we have $w_{s+1} = \lfloor \phi n \rfloor$ if $s$ is odd and $w_{s+1} = \lceil \phi n \rceil$ if $s$ is even.*

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This is true when $s$ is odd for $1 \leq a \leq b \leq f_t$. □

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$$D(n) = \begin{cases} \frac{\sqrt{5}n}{2\phi^{q+1}} + \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 1 \mod 4, \\ 1 - \frac{\sqrt{5}n}{2\phi^{q+1}} - \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 3 \mod 4. \end{cases}$$

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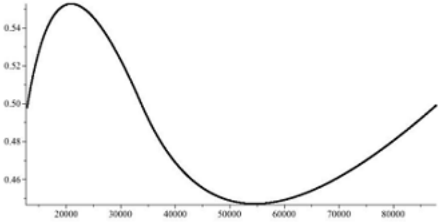
$$D(n) = \begin{cases} \frac{\sqrt{5}n}{2\phi^{q+1}} + \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 1 \mod 4, \\ 1 - \frac{\sqrt{5}n}{2\phi^{q+1}} - \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, \ q \equiv 3 \mod 4. \end{cases}$$

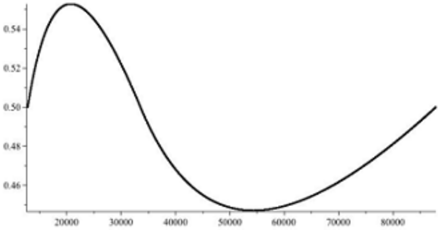
**Proof.**

Count triples $(s, a, b)$ with $s$ odd. □

# Applications




(a) Data plot of $D(n)$.

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We know that $w_{s+1} = \lfloor \phi n \rfloor$ or $w_{s+1} = \lceil \phi n \rceil$. Intuitively, the smaller $\phi n - \lfloor \phi n \rfloor$ is, the more likely it is that $w_{s+1} = \lfloor \phi n \rfloor$.

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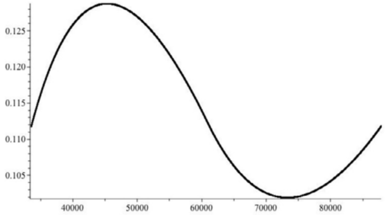
## Theorem (Chung, Graham, S. (2019))

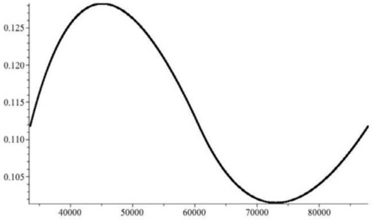
*We have $P(n, r) = 0$ if $r \geq 1 - \frac{1}{\sqrt{5}}\phi^{-1} \approx .72$. Otherwise, given $n$, let $c, p$ be such that $n = \frac{1}{\sqrt{5}}c\phi^p$ with $\frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}}\phi$. Then $P(n, r)$ satisfies*

$$\begin{cases} -\frac{1}{2}\phi^{-1}c + (1-r) + \left(r^2 - r + \frac{1}{2\sqrt{5}}\phi^{-1}\right)c^{-1} + O(n^{-1/2}) & p \text{ odd, } c \leq (1-r)\phi, \\ \frac{\sqrt{5}}{2}\phi\left(r - \frac{1}{\sqrt{5}}\phi\right)^2 c^{-1} + O(n^{-1/2}) & p \text{ odd, } c \geq (1-r)\phi, \\ -\frac{1}{2}c + (1-r) + \left(\phi^{-1}r^2 - \phi^{-1}r + \frac{1}{2\sqrt{5}}\phi^{-2}\right)c^{-1} + O(n^{-1/2}) & p \text{ even, } c \leq 1-r, \\ \frac{\sqrt{5}}{2}\left(r - \frac{1}{\sqrt{5}}\phi\right)^2 c^{-1} + O(n^{-1/2}) & p \text{ even, } 1-r \leq c \leq r \\ \frac{1}{2}c - r + \left(\phi r^2 - \phi r + \frac{1}{2\sqrt{5}}\phi^2\right)c^{-1} + O(n^{-1/2}) & p \text{ even, } c \geq r. \end{cases}$$

# Applications




(a) Data plot of $P(n, .5)$.

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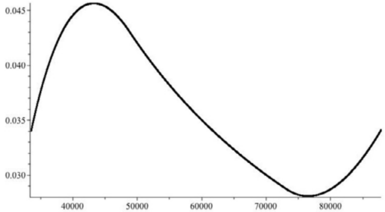
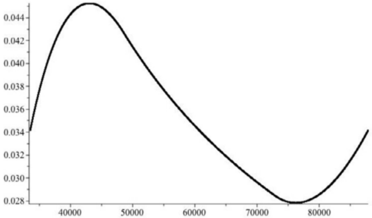



(c) Data plot of $P(n, .6)$.

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Define $g_k$ to be the sequence with

$$g_1 = 1,\ g_2 = \alpha,\ g_{k+2} = \alpha g_{k+1} + \beta g_k.$$

Also define

$$\gamma = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\beta}),\ \lambda = \frac{1}{2}(\alpha - \sqrt{\alpha^2 + 4\beta}).$$

Note that when $\alpha = \beta = 1$ we have $g_k = f_k,\ \gamma = \phi,\ \lambda = -\phi^{-1}$.

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  *and only finitely many $n$ with*

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*Given an integer $p$, let $d$ denote the smallest integer such that $\delta := \beta\gamma^{-1}p - \gamma d \le \alpha$. If $\beta \le p \le \lceil \gamma^2 \rceil - 2$ and $1 \le c \le (p - \beta + 1)\gamma/\alpha$, then*

$$n_{c,r}^{-1}|S_p \cap [n_{c,r}]| = c^{-1}\left(\frac{(2\beta - 2d - 1)\gamma(\alpha - 2\delta + \alpha^{-1}\delta^2)}{2\beta^2(\gamma^2 - 1)} + \frac{\gamma^2}{\gamma^2 - 1}\sum_{q=d+1}^{\beta-1}\frac{\beta - q}{\beta^2}\right) + O(\gamma^{-r} + (\beta\gamma^{-2})^r),$$

*where* $n_{c,r} := \left\lfloor \frac{c\beta}{(\gamma-\lambda)^2}\gamma^{2r+1} \right\rfloor$.

When $\beta = 1$ the proof is essentially the same as before, otherwise one has to be careful about the divisibility condition.

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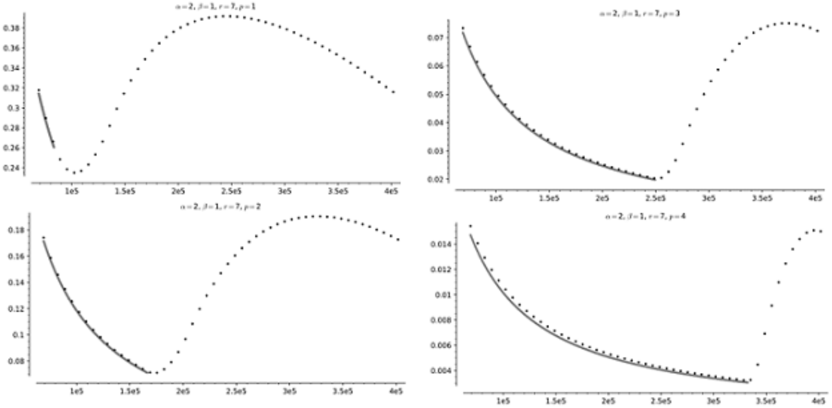
$\alpha = 2,\ \beta = 1,\ p = 1, 2, 3, 4.$



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- *The set of* $n$ *with* $\mathrm{S}(n) = \{(1, 1)\}$ *has density 1.*

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How slow is the slowest slow walk? Define $\mathsf{s}(n) = \max_{(\alpha,\beta)} s^{\alpha,\beta}(n)$, as well as the pairs achieving this $\mathsf{S}(n) = \{(\alpha, \beta) : s^{\alpha,\beta}(n) = \mathsf{s}(n)\}$. A priori, any pair $(\alpha, \beta)$ could be an element of $\mathsf{S}(n)$ for some $n$. However, it turns out that only a finite number of pairs have this property.

**Theorem (S. (2019))**

*Let* $R = \{(1, 1), (1, 2), (1, 3), (2, 1), (1, 4)\}$.

- *For all* $n > 1$*, we have* $\mathsf{S}(n) \subseteq R$.
- *For all* $(\alpha, \beta) \in R$*, there exists an* $n$ *with* $(\alpha, \beta) \in \mathsf{S}(n)$.
- *The set of* $n$ *with* $\mathsf{S}(n) = \{(1, 1)\}$ *has density 1.*

Corollary: the Fibonacci sequence *is* special!

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For the rest of the proof we consider a more general setting: given a set of relatively prime pairs $T$, define
$\mathsf{s}_T(n) = \max_{(\alpha,\beta)\in T} s^{\alpha,\beta}(n)$, $\mathsf{S}_T(n) = \{(\alpha, \beta) : s^{\alpha,\beta}(n) = \mathsf{s}_T(n)\}$.

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*There exists a finite set $R_T$ and number $n_T$ such that $\mathrm{S}_T(n) \subseteq R_T$ for all $n \geq n_T$.*

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## Lemma

*For all $n$ with $s^{\alpha,\beta}(n) > 2$, we have*

$$\frac{1}{2}\log_\gamma(n) - 1 \leq s^{\alpha,\beta}(n) \leq \log_\gamma(n) + 2.$$

The two extreme cases are $n = g_{s-1} + \beta g_{s-2} \approx \gamma^s$ and $n = \beta g_{s-1} g_s \approx \gamma^{2s}$.

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Let $(\alpha', \beta') \in T$ such that $\gamma_{\alpha', \beta'} = \min\{\gamma_{\alpha,\beta} : (\alpha, \beta) \in T\} := \Gamma$.

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If $T$ is every pair, then the only pairs which could be in $R$ are those with $\log_\phi \gamma < 2$. For (1,4) we have $\log_\phi \gamma \approx 1.95$, so it just barely works!

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**Theorem**

*If $T$ is such that there exists a unique pair $(\alpha', \beta') \in T$ with $\gamma_{\alpha',\beta'} = \min_{(\alpha,\beta)} \gamma_{\alpha,\beta}$, then almost every $n$ has $\mathsf{S}_T(n) = \{(\alpha', \beta')\}$.*

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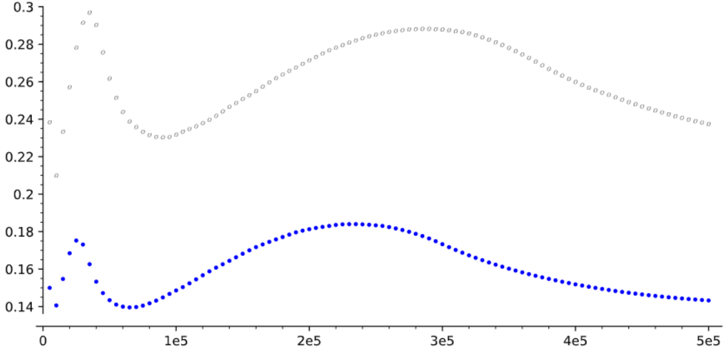
What if there isn't such a unique pair? For example, if $T = \{(1,6),(2,3)\}$ we have $\gamma_{1,6} = \gamma_{2,3} = 3$.

# Slowest Slow Walks

Let $e_{\alpha,\beta}(n) = n^{-1}|\{m : m \leq n, \mathrm{S}_T(m) = \{(\alpha, \beta)\}\}|$ for $T = \{(1, 6), (2, 3)\}$.




Here the open black circles are $e_{2,3}(n)$ and the solid blue dots are $e_{1,6}(n)$.

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- What happens if you require a slow walk to hit two prescribed numbers $n_1$ and $n_2$? Note that $w_1 = n_1,\ w_2 = n_2$ works, so this is well defined.

# The End

Thank You!